Jackson formula

Shaowei Wu

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1 Lorentz transformation

1.1 K' moving at speed v along z direction viewed by K

$$\left.\begin{aligned} x_0' &= \gamma(x_0 - \beta x_1) \\ x_1' &= \gamma(x_1 - \beta x_0) \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} (11.16)$$

where $\beta = |\boldsymbol{\beta}| = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ (11.17)
Inverse Lorentz transformation

$$\left.\begin{aligned} x_0 &= \gamma\left(x_0' + \beta x_1'\right) \\ x_1 &= \gamma\left(x_1' + \beta x_0'\right) \\ x_2 &= x_2' \\ x_3 &= x_3' \end{aligned}\right\} (11.18)$$

Generally,

$$\left.\begin{aligned} x_0' &= \gamma\left(x_0 - \boldsymbol{\beta}\cdot\mathbf{x}\right) \\ \mathbf{x}' &= \mathbf{x} + \tfrac{(\gamma-1)}{\beta^2}(\boldsymbol{\beta}\cdot\mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta}x_0 \end{aligned}\right\} (11.19)$$

1.2 4-vectors

$$(A_0, \boldsymbol{A}) = (A_0, A_1, A_2, A_3)$$

$$\left.\begin{aligned} A_0' &= \gamma\left(A_0 - \boldsymbol{\beta}\cdot\mathbf{A}\right) \\ A_\parallel' &= \gamma\left(A_\parallel - \beta A_0\right) \\ \mathbf{A}_\perp' &= \mathbf{A}_\perp \end{aligned}\right\} (11.22)$$

$\perp$ and $\parallel$ mean perpendicular and parallel to $\boldsymbol{v}$.
The scalar product of two 4-vectors are invariant.
Notation: $(x_0, \boldsymbol{x}) \equiv (x_0 = ct, x_1 = z, x_2 = x, x_3 = y)$

1.3 proper time τ

$$d\tau = dt\sqrt{1-\beta^2(t)} = \frac{dt}{\gamma(t)}$$

(11.26)

$$t_2 - t_1 = \int_{\tau_1}^{\tau_2} \frac{d\tau}{\sqrt{1-\beta^2(\tau)}} = \int_{\tau_1}^{\tau_2} \gamma(\tau) d\tau$$

(11.27)

Timelike:

$$c^2\left(t_1 - t_2\right)^2 - \left|\mathbf{x}_1 - \mathbf{x}_2\right|^2 > 0$$

Spacelike:

$$c^2\left(t_1 - t_2\right)^2 - \left|\mathbf{x}_1 - \mathbf{x}_2\right|^2 < 0$$

Lightlike:

$$c^2\left(t_1 - t_2\right)^2 - \left|\mathbf{x}_1 - \mathbf{x}_2\right|^2 = 0$$

1.4 relativistic Doppler shift

phase of a wave $\phi = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$ is invariant.
$(k_0, \boldsymbol{k}) \equiv (\omega/c, \boldsymbol{k})$ is a 4-vector.
For light waves: $k_0 = |\boldsymbol{k}|$. Thus, $\omega' = \gamma\omega(1 - \beta cos\theta)$ (11.30)

2 addition of velocity

A point P moves in inertia frame K' with velocity $\boldsymbol{u} \equiv \frac{d\mathbf{x}'}{dt'}$. K' moves in inertia frame K with velocity $\boldsymbol{v}$. What is the velocity of P in K?

$$\left.\begin{aligned} u_\parallel &= \frac{u_\parallel' + v}{1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}} \\ \mathbf{u}_\perp &= \frac{\mathbf{u}_\perp'}{\gamma_v\left(1 + \frac{\mathbf{v}\cdot\mathbf{u}'}{c^2}\right)} \end{aligned}\right. \quad (11.31)$$

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel, then:

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}} \quad (11.33)$$

3 relativistic dynamics

3.1 4-velocity $(U_0, \mathbf{U}) = (\gamma_u c, \gamma_u \mathbf{u})$

$$\left.\begin{aligned} U_0 &\equiv \tfrac{dx_0}{d\tau} = \tfrac{dx_0}{dt}\tfrac{dt}{d\tau} = \gamma_u c \\ \mathbf{U} &\equiv \tfrac{d\mathbf{x}}{d\tau} = \tfrac{d\mathbf{x}}{dt}\tfrac{dt}{d\tau} = \gamma_u \mathbf{u} \end{aligned}\right\} (11.36)$$

3.2 4-momentum $(p_0, \mathbf{p}) = (E/c, \gamma_u m\mathbf{u}) = m(U_0, \mathbf{U})$

The only way to construct a 4-vector of momentum is to multiply 4-velocity by rest mass m.
The momentum of a particle of mass m and velocity $\mathbf{u}$ is

$$\mathbf{p} = \gamma_u m\mathbf{u} = \frac{m\mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (11.46)$$

The total energy of a particle of mass m is

$$E = \gamma_u mc^2 = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}} \quad (11.51)$$

Kinetic energy: $T(u) = mc^2(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1)$ (11.49)
The conservation of total energy and total momentum can be written as conservation of total 4-momentum. We also have:

$$E = \sqrt{c^2p^2 + m^2c^4} \quad (11.55)$$

3.3 rapidity/boost parameter ξ

$$\beta = tanh\xi$$

$$\gamma = cosh\xi$$

$$\gamma\beta = sinh\xi$$

(11.20)

4 homogeneous Lorentz group/Poincare group

Mathematical equations expressing the law of nature must be covariant/inariant in form.

4.1 tensor of rank k

Transformation in space time point: $x \to x'$

rank 0: scalar (invariant)

rank 1: contravariant vector

$$A'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial x^{\beta}} A^{\beta} \quad (11.61)$$

rank 1: covariant vector

$$A'_{\alpha} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} A_{\beta} \quad (11.62)$$

rank-2 contravariant tensor

$$F'^{\alpha\beta} = \frac{\partial x'^{\alpha}}{\partial x^{\gamma}} \frac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta} \quad (11.63)$$

rank-2 covariant tensor

$$F'_{\alpha\beta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x^{\delta}}{\partial x'^{\beta}} F_{\gamma\delta} \quad (11.64)$$

inner/scalar product of two vectors:

$$B \cdot A \equiv B_{\alpha} A^{\alpha} \quad (11.65)$$

4.2 metric tensor

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\alpha\beta} \quad (11.81)$$

$$x_{\alpha} = g_{\alpha\beta} x^{\beta} \quad (11.72)$$

$$x^{\alpha} = g^{\alpha\beta} x_{\beta} \quad (11.73)$$

Concisely we have:

$$A^{\alpha} = \left(A^0, \mathbf{A}\right), \quad A_{\alpha} = \left(A^0, -\mathbf{A}\right) \quad (11.75)$$

$$\begin{aligned} \partial^{\alpha} &\equiv \frac{\partial}{\partial x_{\alpha}} = \left(\frac{\partial}{\partial x^0}, -\nabla\right) \\ \partial_{\alpha} &\equiv \frac{\partial}{\partial x^{\alpha}} = \left(\frac{\partial}{\partial x^0}, \nabla\right) \end{aligned} \quad (11.76)$$

$$\partial^{\alpha} A_{\alpha} = \partial_{\alpha} A^{\alpha} = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \mathbf{A} \quad (11.77)$$

Laplacian operator:

$$\Box \equiv \partial_{\alpha} \partial^{\alpha} = \frac{\partial^2}{\partial x^{02}} - \nabla^2 \quad (11.78)$$

5 relativistic electrodynamics

5.1 Lorentz force equation

$$\frac{d\mathbf{p}}{dt} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \quad (11.124)$$

If we use proper time instead, then

$$\begin{aligned} \frac{d\mathbf{p}}{d\tau} &= \frac{d\mathbf{p}}{dt}\frac{dt}{d\tau} \\ &= q\gamma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \\ &= \frac{q}{c}(\gamma c \mathbf{E} + \gamma \mathbf{v} \times B) \\ &= \frac{q}{c}\left(U_0 \mathbf{E} + \mathbf{U} \times \mathbf{B}\right) \end{aligned} \quad (11.125)$$

Similar to Newton dynamic, we have:

$$\frac{dp_0}{d\tau} = \frac{q}{c} \mathbf{U} \cdot \mathbf{E} \quad (11.126)$$

5.2 continuity equation

Electric charge is conserved. The continuity equation still works.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \quad (11.127)$$

Equation (11.77) and (11.127) implies that we can write charge density ρ and current density J in 4-vector form:

$$J^{\alpha} = (c\rho, \mathbf{J}) \quad (11.128)$$

Then we have:

$$\partial_{\alpha} J^{\alpha} = 0 \quad (11.129)$$

5.3 Lorentz gauge

$$\begin{aligned} \tfrac{1}{c^2}\tfrac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} &= \tfrac{4\pi}{c}\mathbf{J} \\ \tfrac{1}{c^2}\tfrac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi &= 4\pi\rho \end{aligned} \quad (11.130)$$

Lorentz condition:

$$\frac{1}{c}\frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} = 0 \quad (11.131)$$

$$A^{\alpha} = (\Phi, \mathbf{A}) \quad (11.132)$$

Equation (11.130)+(11.78):

$$\Box A^{\alpha} = \frac{4\pi}{c} J^{\alpha} \quad (11.133)$$

Equation (11.131)+(11.77):

$$\partial_{\alpha} A^{\alpha} = 0 \quad (11.133)$$

Field in terms of potential:

$$\begin{aligned} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\Phi \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned} \quad (11.134)$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\frac{dp^{\alpha}}{d\tau} = m\frac{dU^{\alpha}}{d\tau} = \frac{q}{c} F^{\alpha\beta} U_{\beta}$$

5.4 transformation of electromagnetic field

Transformation of the rank-2 tensor $F^{\alpha\beta}$ follows equation (11.63). Thus we can deduce the transformation of field directly as below.

$$\begin{array}{ll} E_1' = E_1 & B_1' = B_1 \\ E_2' = \gamma\left(E_2 - \beta B_3\right) & B_2' = \gamma\left(B_2 + \beta E_3\right) \\ E_3' = \gamma\left(E_3 + \beta B_2\right) & B_3' = \gamma\left(B_3 - \beta E_2\right) \end{array} \tag{11.148}$$

$$\begin{array}{l} \mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) \\ \mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) \end{array} \tag{11.149}$$

For electromagnetic field transformation of a charged particle, see Jackson 11.152.